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Solution of the Equations of Motion for Einstein's Field in Fractional *D* Dimensional Space-Time

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Abstract As a continuation of Sadallah et al. work (M. Sadallah, S. Muslih and D. Baleanu, Equations of motion for Einstein's field in non-integer dimensional space. Czechoslov. J. Phys. 56:323, 2006), the fractional action function S is given as an integration over fractional spatial dimension D_s and fractional time D_t dimension. The variational principle which minimize S leads to Euler-Lagrange equations of motion in $D_s + D_t$ fractional dimensions. As an example we extend our study to obtain the equations of motion for Einstein's field in fractional $D_s + D_t$ fractional dimensions of $N + 1$ space-time coordinates. It is shown that the time dependent solutions are single valued for only $D_s = 4$ dimensional space. Also the angular solutions are convergent for any value of D_s.

Keywords Conformal scalar field · Einstein's universe · Fractional space

1 Introduction

In 1918 the Mathematician Felix Hausdorff introduced the notion of fractional dimension. This concept became very important especially after the revolutionary discovery of fractal geometry by Mandelbrot [1], where he used the concept of fractionality and worked out the relations between fractional dimension and integer dimension by using the scale method i.e. $d^\alpha x = \frac{\pi^{\alpha/2}|x|^{\alpha-1}}{\Gamma(\alpha/2)}dx$, $0 < \alpha \leq 1$. And numerous efforts has been made by researchers in various branches of science and technology [2–21]. Besides, there are other approaches to describe fractional dimension. These include, fractional calculus (a generalization of differentiation and integration to non integer order) [22] and the analytic continuation of the dimension in Gaussian integral [13, 14, 23–25]. The later is often used in

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quantum field theory [24, 25], and introduced in the dimensional regularization method, $\int f(x)d^n x = \frac{2\pi^{(n)/2}}{\Gamma(\frac{n}{2})} \int_0^\infty f(x)x^{n-1}dx$ (a method of removing the divergent term in the evaluation of Feynman diagram term in order to avoid the loop divergence).

Historically, the first example of fractional physical objects was the Brownian motion [1]. In quantum physics the first successful attempt applying of fractality concept was Feynman path integral approach [26], where Feynman and Hibbs [26] reformulated the non- relativistic quantum mechanics as a path integral over Brownian paths.

There are much interest to study the equations of motion in non-integer dimensional space [27–39]. He [27, 28], has proposed a method to replace the real confining structure with an effective space, where the measure of its anisotropy is given by non-integer dimensions.

Recently, [29] the Euler-Lagrange equations of motion in non-integer dimensions and solutions of Schrödinger equation are obtained in three variables system.

Another progress in this field was obtained in [40], where the fractional Euler-Lagrange equations of motion for classical fields are obtained and a generalization to the simplest variational problem as proposed in reference [41] is given. Besides, using the scaling concepts of Mandelbrot [1], it is possible to re-write the dimensional regularization as an integration over the fractional dimensional space. This allows Muslih and Agrawal [42] to introduce new technique to solve the non-homogenous fractional differential equations of motion by re-defining the integral transformed method as a fractional spatial-time integral transformed method.

For the above mentioned reasons obtaining the Euler-Lagrange equations of motion for Einstein's field in fractional dimensions is an interesting issue.

The plan of the paper is as follows: in Sect. 2 the Euler-Lagrange equations in non-integer dimensions are reviewed. In Sect. 3 the scalar field equation in $D = D_s + D_t$ dimensional fractional space-time is analyzed. Finally, the Sect. 4 is dedicated to our conclusions.

2 Euler-Lagrange Equations in $D_s + D_t$ Dimensional Fractional Space-Time

In this section we shall give a brief review on the Euler-Lagrange equations of motion for fields in non-integer dimensions [29].

A covariant form of the action would involve a Lagrangian density $\mathcal{L}$ via $S = \int_{\partial\Omega'} \mathcal{L} d^{D+1}x = \int \mathcal{L} d^D x dt$ where $\partial\Omega'$ is the boundary for all coordinates. The Lagrangian density $\mathcal{L}$ is defined as, $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi)$ and with $L = \int \mathcal{L} d^D x$. The corresponding covariant Euler-Lagrange equations are

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0, \tag{1}$$

where ϕ is the field variable and ∂_μ is space and time derivative.

For non-integer space-time coordinates, the action function for N degrees of freedom is

$$\begin{aligned} S &= \int_{\partial\Omega'} d^{D_t}t\, d^{D_s}x \mathcal{L}(\phi, \partial_\mu\phi)) \\ &= \int d^{\alpha_t}t \int \prod_{i=1}^{N} d^{\alpha_i}x_i \mathcal{L}(\phi, \partial_\mu\phi), \end{aligned} \tag{2}$$

where, ϕ and $\partial_\mu\phi$ are functions of $(t, x^1, \ldots, x^N)$ and $\partial_\mu = (\frac{\partial}{\partial t}, \frac{\partial}{\partial x^i})$, with i is running from 1 to N and the fractional volume element $d^D x$ and the fractional line element are given respectively as [1, 9, 42]

$$d^{D_s}x = \prod_i d^{\alpha_i}x_i, \tag{3}$$

$$d^{\alpha_i}x_i = \frac{\pi^{\alpha_i/2}|x|^{\alpha_i-1}}{\Gamma(\alpha_i/2)}dx_i, \tag{4}$$

and $D_s = \sum_{i=1}^N \alpha_i$, $D_t = \alpha_t$. In this paper we will consider the limits of $\alpha_\mu = (\alpha_t, \alpha_s)$ as $0 < \alpha_\mu \leq 1$, such that $0 < D \leq N + 1$. We set $\delta S = 0$, it follows that the Euler-Lagrange equations of motion in non-integer dimensions is given by [29]

$$\frac{\partial \mathcal{L}(\phi, \partial_\mu \phi)}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}(\phi, \partial_\mu \phi)}{\partial(\partial_\mu \phi)} - (\alpha_{\mu\nu} - \delta_{\mu\nu})(x^{(-1)})^\nu \frac{\partial \mathcal{L}(\phi, \partial_\mu \phi)}{\partial(\partial_\mu \phi)} = 0, \tag{5}$$

with $\delta_{\mu\nu}$ is a diagonal unit matrix, $(x^{-1}) = column(t^{-1}, (x^{-1}), \ldots, (x^{-N}))$, and $\alpha_{\mu\nu}$ are the diagonal elements of a matrix which include both time and spatial dimensions ($\alpha = dimension(\alpha_t, \alpha_1, \ldots, \alpha_N)$), the spatial dimension of the system is specified by $D_s = Tr(\alpha) - \alpha_t$.

3 Scalar Field Equation in $D_s + D_t$ Dimensional Fractional Space-Time

The Lagrangian density for Einstein's field is given by [43]

$$\mathcal{L} = \frac{1}{2}(-g)^{1/2}\left(g^{\mu\nu}\partial_\mu \Psi \partial_\nu \Psi - \frac{1}{r^2}\Psi^2\right), \tag{6}$$

where g is the Friedmann metric for closed universe and $\mu,\ \nu = 0, 1 \ldots, N$.

The case of $\alpha_t = 1$ is considered in our calculations which means that their is no time-decaying friction term. In this case the Euler-Lagrange equation of motion (5) for this system reads as

$$\nabla^2 \Psi - \frac{\partial^2}{\partial t^2}\Psi - \frac{\Psi}{r^2} + (\alpha_{ij} - \delta_{ij})(x^{-1})^j \partial_j \Psi = 0, \quad i,\ j = 1, \ldots, N, \tag{7}$$

where r represents the Einstein's universe radius and ∇^2 is the Laplacian in N dimensional coordinates $(x_1, x_2, \ldots, x_N)$.

The case for $N = 4$ was considered in reference [44], now let us analyze the Einstein's field equation for any N. Equation (7), reads as

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\alpha_1 - 1}{x_1}\frac{\partial}{\partial x_1} + \frac{\partial^2}{\partial x_2^2} + \frac{\alpha_2 - 1}{x_2}\frac{\partial}{\partial x_2} + \cdots + \frac{\partial^2}{\partial x_N^2} + \frac{\alpha_N - 1}{x_N}\frac{\partial}{\partial x_N}\right)\Psi - \frac{\partial^2}{\partial t^2}\Psi - \frac{\Psi}{r^2} = 0. \tag{8}$$

Choosing α_N as the single parameter for the non-integer dimensions with $\alpha_1 = \alpha_2 = \cdots = \alpha_{N-1} = 1$, so $D_s = \alpha_N + (N - 1)$. In this case equation (8) can be put in the form

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_{N-1}^2} + \frac{D_s - N}{x_N}\frac{\partial}{\partial x_N} + \frac{\partial^2}{\partial x_N^2}\right)\Psi - \frac{\partial^2}{\partial t^2}\Psi - \frac{\Psi}{r^2} = 0. \tag{9}$$

Let us consider the following N dimensional polar coordinates [45, Vol. II, Chap. IX]:

$$\left.\begin{array}{c} x_1 = r\cos\theta_1 \\ x_2 = r\sin\theta_1\cos\theta_2 \\ x_3 = r\sin\theta_1\sin\theta_2\cos\theta_3 \\ \cdots \\ x_{N-1} = r\sin\theta_1\sin\theta_2\sin\theta_3\cdots\sin\theta_{N-2}\cos\theta_{N-1} \\ x_N = r\sin\theta_1\sin\theta_2\sin\theta_3\cdots\sin\theta_{N-1} \end{array}\right\}, \tag{10}$$

where, $r = (\sum_{k=1}^{N} x_k^2)^{1/2} = \text{const.}$, $0 \le \theta_a \le \pi$, $(a = 1, 2, \ldots, N-2)$, $0 \le \theta_{N-2} \le 2\pi$ of R^N.

Equation (9) becomes

$$\begin{aligned} &\frac{1}{r^2}\left(\frac{\partial^2\Psi}{\partial\theta_1^2} + \frac{D_s-2}{\tan\theta_1}\frac{\partial\Psi}{\partial\theta 1}\right) + \frac{1}{r^2\sin^2\theta_1}\left(\frac{\partial^2\Psi}{\partial\theta_2^2} + \frac{D_s-3}{\tan\theta_2}\frac{\partial\Psi}{\partial\theta_2}\right) \\ &\quad + \frac{1}{r^2\sin^2\theta_1\sin^2\theta_2}\left(\frac{\partial^2\Psi}{\partial\theta_3^2} + \frac{D_s-4}{\tan\theta_3}\frac{\partial\Psi}{\partial\theta_3}\right) \\ &\quad + \cdots + \frac{1}{r^2\sin^2\theta_1\sin^2\theta_2\cdots\sin^2\theta_{N-2}}\left(\frac{\partial^2\Psi}{\partial\theta_{N-1}^2} + \frac{D_s-N}{\tan\theta_{N-1}}\frac{\partial\Psi}{\partial\theta_{N-1}}\right) \\ &\quad - \frac{\partial^2}{\partial t^2}\Psi - \frac{\Psi}{r^2} = 0. \end{aligned} \tag{11}$$

Let us assume that (11) is separable. For this case Ψ is given by

$$\Psi(r = \text{const.}, \theta_1, \theta_2, \ldots, \theta_{N-1}) = R(r)T(t)\prod_{i=1}^{N-1}\Theta_i(\theta_i), \tag{12}$$

having in mind that for Einstein's universe r is constant and assuming $T(t) = e^{-i\nu t}$, (11) reduces to the following $(N-1)$ angular separable equations

$$\left(\frac{d^2\Theta_1}{d\theta_1^2} + \frac{D_s-2}{\tan\theta_1}\frac{d\Theta_1}{d\theta_1}\right) + ((\nu^2r^2-1)\sin^2\theta_1 - \lambda_1)\Theta_1 = 0, \tag{13}$$

$$\left(\frac{d^2\Theta_2}{d\theta_2^2} + \frac{D_s-3}{\tan\theta_2}\frac{d\Theta_2}{d\theta_2}\right) + \left(\lambda_1 - \frac{\lambda_2}{\sin^2\theta_2}\right)\Theta_2 = 0, \tag{14}$$

$$\left(\frac{d^2\Theta_3}{d\theta_3^2} + \frac{D_s-4}{\tan\theta_3}\frac{d\Theta_3}{d\theta_3}\right) + \left(\lambda_2 - \frac{\lambda_3}{\sin^2\theta_3}\right)\Theta_3 = 0, \tag{15}$$

$\cdots$

$$\left(\frac{d^2\Theta_{N-2}}{d\theta_{N-2}^2} + \frac{D_s-(N-1)}{\tan\theta_{N-2}}\frac{d\Theta_{N-2}}{d\theta_{N-2}}\right) + \left(\lambda_{N-3} - \frac{\lambda_{N-2}}{\sin^2\theta_{N-2}}\right)\Theta_{N-2} = 0, \tag{16}$$

$$\left(\frac{d^2\Theta_{N-1}}{d\theta_{N-1}^2} + \frac{D_s-N}{\tan\theta_{N-1}}\frac{d\Theta_{N-1}}{d\theta_{N-1}}\right) + \lambda_{N-2}\Theta_{N-1} = 0. \tag{17}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{N-2}$ are separation constants.

Equation (13) could be solved by using the following change of variables

$$x = \cos\theta_1, \quad \Theta_1(\theta_1) \to X(x), \tag{18}$$

in this case (13), reads as

$$(1-x^2)^2\frac{d^2X}{dx^2} - (D_s-1)x(1-x^2)\frac{dX}{dx} + \left[(\nu^2r^2-1)(1-x^2) - \lambda_1\right]X = 0. \tag{19}$$

Following the series solution of the following second order ordinary differential equation [29, 44]

$$(1-x^2)\frac{d^2X}{dx^2} - 2(\lambda+1)x\frac{dX}{dx} + \left[n(n+2\lambda+1) - \frac{l(l+2\lambda)}{(1-x^2)}\right]X = 0. \tag{20}$$

We can show that this equation is convergent if $n = 0, 1, \ldots,$ and $l = 0, 1, \ldots,$, where $l \le n$ and its solution is given by following solution [29, 44]

$$X^{\lambda}_{n,l} = (1-x^2)^{l/2}T^{\lambda+l}_{n-l}(x), \tag{21}$$

where $T^{\lambda+l}_{n-l}(x)$ is a Gegenbauer polynomial [29, 44, 46].

Now, a series solution of (19) is convergent if [44]

$$\lambda_1 = l(l+D_s-3), \quad l = 0, 1, \ldots, \tag{22}$$

and

$$\nu^2r^2 - 1 = n(n+D_s-2), \quad n = 0, 1, 2, \ldots, \ l \le n, \tag{23}$$

with a solution as

$$\Theta_1{}^{D}_{n,l}(\cos\theta_1) = (1-\cos^2\theta_1)^{l/2}T^{(D_s-3)/2+l}_{n-l}(\cos\theta_1), \tag{24}$$

we obtain the photon scalar frequency as

$$\nu = \frac{\sqrt{n(n+D_s-2)+1}}{r}, \quad n = 0, 1, 2, \ldots. \tag{25}$$

The time dependent solution $T(t) = e^{-i\nu t}$, is single valued if and only if

$$\nu = \frac{n}{r}, \quad n = 1, 2, 3, \ldots. \tag{26}$$

Equations (24) and (25) lead us to following condition

$$D_s = 4, \tag{27}$$

Again by the aid of (20), we arrive at the convergent solutions of (14–17) if and only if $\lambda_2, \lambda_3, \ldots, \lambda_{N-3}, \lambda_{N-2}$ are given by

$$\lambda_2 = m_1(m_1+D_s-4), \quad m_1 = 0, 1, 2, \ldots, \ m_1 \le l, \tag{28}$$

$$\lambda_3 = m_2(m_2+D_s-5), \quad m_2 = 0, 1, 2, \ldots, \ m_2 \le m_1, \tag{29}$$

(30)

$$\dots \tag{31}$$

$$\lambda_{N-3} = m_{N-4}(m_{N-4} + D_s - N + 1), \quad m_{N-4} = 0, 1, 2, \dots, \; m_{N-4} \le m_{N-5}, \tag{32}$$

$$\lambda_{N-2} = m_{N-3}(m_{N-3} + D_s - N), \quad m_{N-3} = 0, 1, 2, \dots, \; m_{N-3} \le m_{N-4}. \tag{33}$$

It is obvious that the generalized quantum number $n, l, m_1, m_2, \dots, m_{N-2}$, obey the following quantization rule

$$m_{N-3} \le m_{N-4} \le \cdots \le m_1 \le l \le n, \tag{34}$$

with the following solutions

$$\Theta_{N-2}(\cos\theta_{N-2}) = (1 - \cos^2\theta_{N-2})^{m_{N-3}/2} T_{m_{N-4}-m_{N-3}}^{(D_s-N+1)/2+m_{N-3}}(\cos\theta_{N-2}), \tag{35}$$

$$\Theta_{N-1}(\cos\theta_{N-1}) = T_{m_{N-3}}^{(D_s-N-1)/2}(\cos\theta_{N-1}), \quad N \ge 4. \tag{36}$$

One should notice that the angular solutions (24, 35, 36) are convergent for any value of D_s.

4 Conclusions

Using the concepts of Mandelbrot [1] of scaling the space-time dimensions, the equations of motion (5) are obtained in fractional space-time dimensions with an extra new space-time dependent damping term $(\alpha_{\mu\nu} - \delta_{\mu\nu})(x^{(-1)})^{\nu} \frac{\partial \mathcal{L}(\phi, \partial_\mu \phi)}{\partial(\partial_\mu \phi)}$. As an example have introduced the solution of Euler-Lagrange equation of motion for massless conformal scalars field in an Einstein's universe with a non integer dimensional space. Using the convergence conditions on the series solution of the angular equations (13–17), we obtained the separation constants $\lambda_1, \; \lambda_2, \dots, \lambda_{N-2}$ and also the eigen frequencies ν. It is observed that the $T(t)$ has physical solutions (single valued) if and only if $D_s = 4$. Besides the angular solutions are convergent for any value of D_s. This behavior of our new solutions is due to irregularity of the objects and this introduce the fractionality in the space which is related to the roughness of the materials.

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